A direct solution of allocation problems

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This paper deals with tree-structured networks that interconnect n *given points* $A_1, \ldots, A_n$ *in the plane through up to* $(n - 2)$ *auxiliary junction points* $S_1, \ldots, S_{n-2}$. *The problem of locating these auxiliary vertices is tackled to minimize the total cost of the connection network. The formulation of the problem is not bound by the condition that specific link costs are equal for all branches, in order to comprise a broader range of applications, such as distribution networks. Starting from a well-known mechanical analogy, the authors arrive at some mathematical expressions and conditions of their applicability that allow calculation of the optimal coordinates of the unknown* S_i *points directly without resorting to the usual iterative procedures. The method is illustrated for the case of* $n = 3$ *points and its extension to more general cases is presented. The formulas obtained also apply to the particular case of determining Steiner minimal trees, specific link cost being constant. An example of application to electric distribution networks is also given.*

Keywords: tree networks, auxiliary junctions, optimum location, mathematical model

Introduction

In designing, one often meets with the problem of optimizing a system that can be represented as a network interconnecting a set of given points. This can be a transport, communication, or distribution network serving a certain number of user points (or producer points) with known characteristics (location, requirements, production capacity, etc.). The objective function to be minimized is almost always total network cost.

Should the connection network be tree-structured, the introduction of auxiliary junction nodes that do not coincide with any of the fixed points can be advantageous. The objective in this case consists in the optimum location of such auxiliary points.

This problem, known in its simplest formulation as the "Steiner problem," has been attacked by A. Weber for connection networks whose unit link costs are constant over the whole plane examined. The optimum solution in this instance is represented by the network with shortest total link length.[1]

Figure 1 gives an example of a network connecting three points A_1, A_2, A_3 with the auxiliary point S (Steiner point) that minimizes total link length. The position of the solution point S is the one that allows the three sides of the triangle $(A_1A_2A_3)$ to be seen under the same angle, that is, 120°.

For networks with n points ($n > 3$) it can be shown that the number of auxiliary junctions with which the minimal cost tree is obtained never exceeds $(n - 2)$. Moreover, at each junction introduced, three and only three branches must converge. If cost is a function of length alone, all adjacent branches must form angles of 120° with each other, whereas in the general case the angles are different. The number of branches is always equal to the number of total points (n plus the Steiner points) less one.

Given a certain number n and a certain distribution of allocated points, many different Steiner trees exist that enjoy the above properties. Some of these trees, called "locally" minimal solutions, cannot be made shorter by a slight perturbation, but not all Steiner trees that are locally minimal are solutions of minimal length.

To find a minimal network, a systematic search can be made among the Steiner locally minimal trees and the shortest one chosen. Melzak[2] was the first to develop an algorithm for the Steiner problem, but even for simple problems, computation time can be excessive because of the enormous number of possibilities examined. Various more efficient methods have been proposed that prune those processing stages that might provide only networks of considerable length. The latest pruning techniques are much less computational intensive: a program developed by E. J. Cockayne and D. E. Hewgill of Victoria University solves in a few minutes all problems with 17 points and the majority of those with 30 random points. All these programs incorporate constructions similar to that used for solving the problem with three points.

If the costs are not just a function of link length alone, then the research of optimum location of the auxiliary junctions is more complex. The algorithms proposed in literature are, to the authors' knowledge, exclusively of the iterative and approximate type.[3–8]

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Received 20 July 1987; accepted 21 December 1990

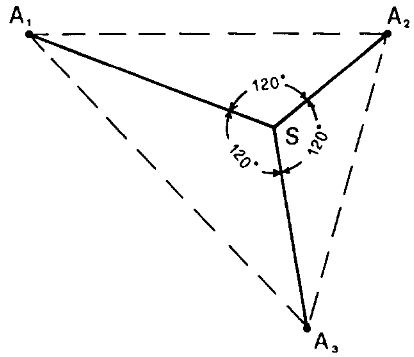

Figure 1. Minimum-length network connecting three points

The present work provides, as an effective alternative, the direct analytical solution to the problem of locating, for each Steiner topology examined, the locally minimal tree in the most general case of link costs varying with both distance and user points requirements. The minimal tree in absolute can then be obtained by comparing the locally minimal trees. This analysis may be conducted following the same approach and similar pruning methods as those used in the search for the minimal length tree.

Formulation of the problem

The network cost minimization problem is of the continuous type and can be stated as follows: Given n points $A_1, \ldots, A_n$ in the plane with given coordinates and characteristics, we want to locate at most $(n-2)$ auxiliary junction nodes S_j at any point on the plane such that the connection network costs are minimized.

To solve the problem for the most general case, we have to first consider the elementary case of a network joining three points by means of a single auxiliary junction node.

Solution of the case $n = 3$

Referring to *Figure 2*, let A_1, A_2, and A_3 be three assigned points to be linked through a tree network whose total cost is minimized by introducing an auxiliary junction point S, with x_s, y_s unknown coordinates.

We denote with (x_1, y_1), (x_2, y_2), (x_3, y_3) the coordinates of the points A_1, A_2, A_3; w_1, w_2, and w_3 are the costs per unit length of the three network branches SA_1, SA_2, SA_3.

The unit costs w_1, w_2, w_3 are usually determined on the basis of the requirements and characteristics of the three points to be linked.

Moreover let l_1, l_2, l_3 denote the unknown lengths of the three network branches SA_1, SA_2, SA_3.

The mathematical solution found by the authors in this work departs from a known mechanical analogy of the problem; in fact, the optimization problem can be represented by the function

$$\min\,[w_1l_1 + w_2l_2 + w_3l_3] \qquad (1)$$

where the unit costs w_1, w_2, w_3 are known quantities and are independent of the position of the junction point S.

In correspondence with the minimum cost function we have

$$w_1dl_1 + w_2dl_2 + w_3dl_3 = 0 \qquad (2)$$

where dl_1, dl_2, dl_3 are the congruent variations in lengths l_1, l_2, l_3 for a generic infinitesimal deviation of S from the minimum cost position.

The expression (2) is formally analogous to the one obtainable by applying the "virtual work" principle to a deformable mechanical system in equilibrium, consisting of three forces w_1, w_2, w_3 converging at the same point S and passing through A_1, A_2, A_3, respectively.[9]

The solution of the problem can therefore be reduced to the location of the junction and equilibrium point of three given forces.

Geometrical solution

Figure 2 shows a geometric construction that leads to the optimum location of the point S that we are seeking.

The triangle (A_1A_2V), called the "weight triangle," is constructed upon the segment A_1A_2 such that its sides A_1A_2, A_2V, VA_1 are proportional to the three cost weights w_3, w_1, w_2 and such that it lies in the semiplane delineated by the straight line t passing through A_1 and A_2 and not containing the point A_3.[7,8]

For the equilibrium of the three forces, the angles under which the segments A_1A_2, A_1A_3, and A_2A_3 are seen by the unknown point S are determined *a priori*; denoting with α_1 and α_2 the angles at the vertices A_1 and A_2 of the weight triangle and generalizing Steiner's theorem, which for forces of equal strengths leads to the 120° rule, we should have

$$\begin{aligned} A_1\hat{S}A_2 &= \alpha_1 + \alpha_2 \\ A_1\hat{S}A_3 &= 180^\circ - \alpha_2 \\ A_2\hat{S}A_3 &= 180^\circ - \alpha_1 \end{aligned} \qquad (3)$$

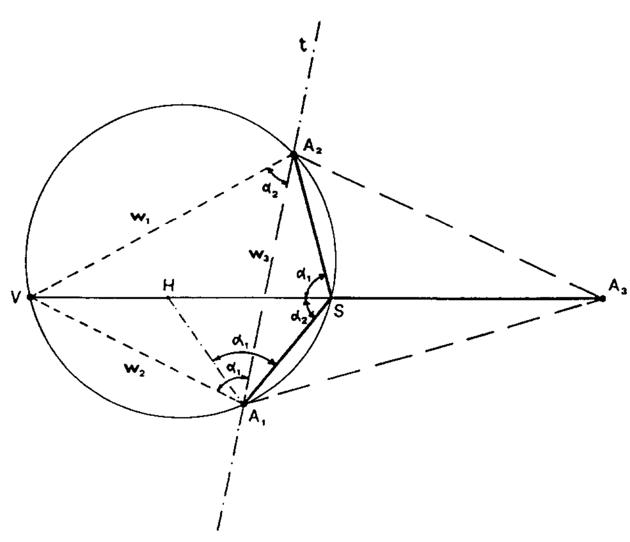

Figure 2. Geometrical optimum location of point S

As can be seen from *Figure 2*, the only point S that satisfies the conditions (3) coincides with the intersection, if it exists, between the segment VA_3 and the arc $\widehat{A_1SA_2}$ of the circle circumscribing the weight triangle. Such an intersection can exist and be distinct from the points A_1, A_2, and A_3 only when the interior angles of the triangle $(A_1A_2A_3)$ satisfy the three conditions

$$\begin{aligned} A_3\hat{A}_1A_2 &< 180° - \alpha_1 \\ A_1\hat{A}_2A_3 &< 180° - \alpha_2 \\ A_2\hat{A}_3A_1 &< \alpha_1 + \alpha_2 \end{aligned} \tag{4}$$

If all the inequalities (4) are verified, then the point S that we are seeking is located inside the triangle $(A_1A_2A_3)$ and is therefore distinct from its vertices; should this not be the case, then the point S is assumed to coincide with that of the three vertices A_1, A_2, or A_3 owing to which one of the conditions (4) is not satisfied. As can be observed, if one of the three conditions (4) is not verified, the other two will certainly be satisfied.

Mathematical solution

The mathematical procedure proposed here, which departs from the mechanical analogy formulation and exploits the properties of the geometric construction given in *Figure 2*, leads to an expression that can be used directly for finding the point S. This point, whose coordinates x_s, y_s are unknown, must belong to the three straight lines that pass through the points A_1, A_2, and A_3 and that form with each other the known angles referred to in (3). The problem therefore reduces to solving the following system of nonlinear equations:

$$\begin{aligned} y_s &= (n_1/d_1)(x_s - x_1) + y_1 \\ y_s &= (n_2/d_2)(x_s - x_2) + y_2 \\ y_s &= (n_3/d_3)(x_s - x_3) + y_3 \\ \frac{(n_1/d_1) - (n_2/d_2)}{1 + (n_1/d_1)(n_2/d_2)} &= -rtg(\alpha_1 + \alpha_2) \\ \frac{(n_1/d_1) - (n_3/d_3)}{1 + (n_1/d_1)(n_3/d_3)} &= -rtg\alpha_2 \end{aligned} \tag{5}$$

where (n_1/d_1), (n_2/d_2), (n_3/d_3) are the unknown slope factors of the straight lines passing through S and the points A_1, A_2, A_3 and r is a "rotation index" of value $+1$ if the perimeter of the triangle of the given points runs in the order A_1-A_2-A_3 in a counterclockwise direction and -1 if it runs in a clockwise one.

The rotation index can be derived analytically once the coordinates of the points A_1, A_2, and A_3 are known:

$$\begin{aligned} r &= k/|k| \\ k &= (y_3 - y_1)(x_2 - x_1) - (y_2 - y_1)(x_3 - x_1) \end{aligned} \tag{6}$$

When the weight triangle exists and the conditions (4) are satisfied, the system (5) always admits a unique solution, that is,

$$\begin{aligned} x_s &= \frac{n_2d_1x_2 + d_1d_2(y_1 - y_2) - n_1d_2x_1}{n_2d_1 - n_1d_2} \\ y_s &= \frac{n_2d_1y_1 - n_1n_2(x_1 - x_2) - n_1d_2y_2}{n_2d_1 - n_1d_2} \end{aligned} \tag{7}$$

in which we first estimate

$$\begin{aligned} n_1 &= rc_3s_2(y_2 - y_1) + rc_2s_3(y_1 - y_3) \\ &\quad + s_2s_3(x_3 - x_2) \\ d_1 &= rc_3s_2(x_2 - x_1) + rc_2s_3(x_1 - x_3) \\ &\quad - s_2s_3(y_3 - y_2) \\ n_2 &= n_1c_3 + rs_3d_1 \\ d_2 &= d_1c_3 - rs_3n_1 \end{aligned} \tag{8}$$

where the following known factors appear:

$$\begin{aligned} c_1 &= \cos\alpha_1 = \frac{w_2^2 + w_3^2 - w_1^2}{2w_2w_3} \\ s_1 &= \sin\alpha_1 = \sqrt{1 - c_1^2} \\ c_2 &= \cos\alpha_2 = \frac{w_1^2 + w_3^2 - w_2^2}{2w_1w_3} \\ s_2 &= \sin\alpha_2 = \sqrt{1 - c_2^2} \\ c_3 &= \cos(\alpha_1 + \alpha_2) = c_1c_2 - s_1s_2 \\ s_3 &= \sin(\alpha_1 + \alpha_2) = \sqrt{1 - c_3^2} \end{aligned} \tag{9}$$

With the expressions (7) the coordinates of point S that we are seeking can be immediately found. *Figure 3* gives the logical flow diagram of the solution algorithm for the case of three fixed points.

Of the six conditional tests illustrated in the flow diagram the first three verify the existence of the weight triangle, that is, if equilibrium of the three forces with strengths equal to the unit costs w_1, w_2, w_3 is possible. This condition is satisfied if, using (9), we get

$$|c_1| \le 1 \qquad |c_2| \le 1 \qquad |c_3| \le 1 \tag{6'}$$

The other three tests relate to the verification of the conditions (4) on the advantage of introducing an auxiliary junction point S distinct from the three fixed points A_1, A_2, A_3.

Should any of the six tests give a negative result, then the introduction of an auxiliary point is uneconomical. In this case, following the diagram of *Figure 3*, we can find which of the trees based on direct links between the given points is the most suitable.

Geometric cost representation: Equivalent user

With the construction proposed in *Figure 2* the total optimized network cost can be evaluated geometrically.[10]

Consider the cost function C_g written as

$$C_g = w_3\left(\frac{w_1}{w_3}l_1 + \frac{w_2}{w_3}l_2 + l_3\right) \tag{10}$$

and identify the point H on the segment VA_3 such that

$$H\hat{A}_1S = \alpha_1$$

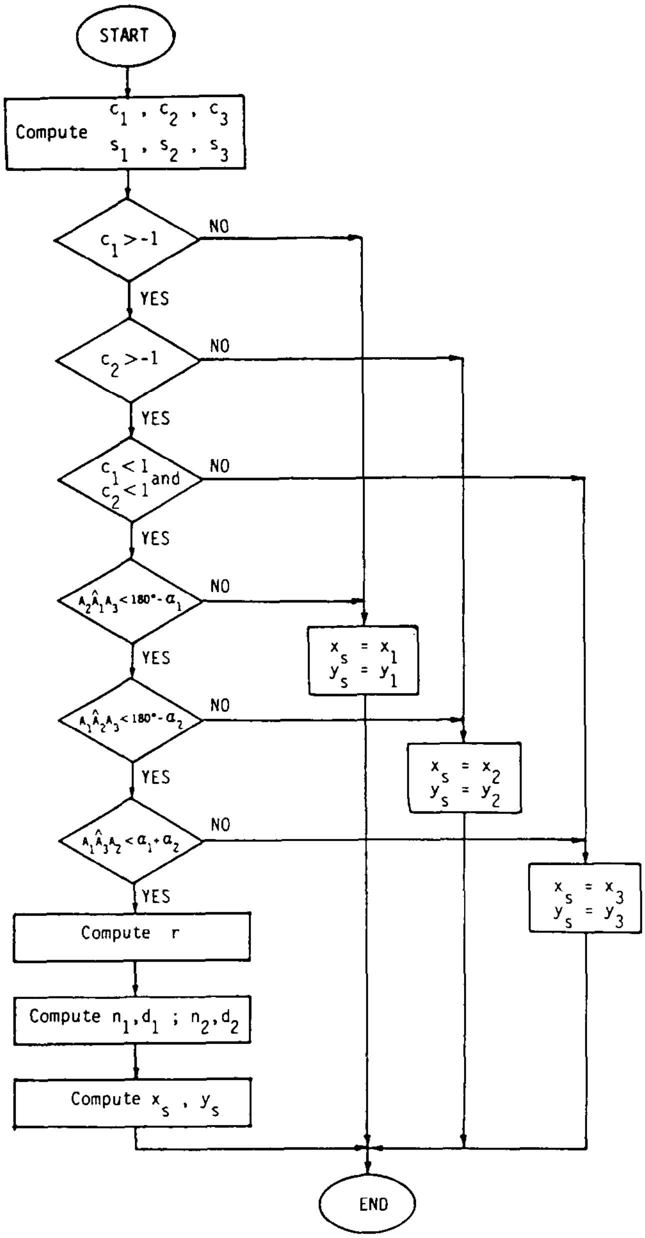

Figure 3. Flowchart of the solution algorithm for the case $n = 3$

Observing the similarity between the triangles (A_1HS) and (A_1VA_2) and that between (A_1HV) and (A_1SA_2), and assuming for simplicity a unitary length scale, we can rewrite (10) as

$$C_g = w_3(\overline{HS} + \overline{VH} + \overline{SA}_3) = w_3\overline{VA}_3 \tag{11}$$

The above expression shows that with the appropriate choice of cost representation scale the segment VA_3 represents the total cost C_g of the optimized network. In the specific case of equal unit costs its length is the same as the total length of the minimal tree.

Moreover, observing that C_g can also be written as

$$C_g = w_1\overline{SA}_1 + w_2\overline{SA}_2 + w_3\overline{SA}_3 \tag{12}$$

and by comparing (11) and (12) it results in

$$w_1\overline{SA}_1 + w_2\overline{SA}_2 = w_3\overline{VS} \tag{13}$$

This expression establishes cost equivalence between the set of the two optimum links SA_1, SA_2 and the single link VS, estimated with the same cost per unit length as that w_3 of the third branch SA_3 with which it is aligned.

On the grounds of the above statements, to the vertex V of the weight triangle we attribute the meaning of "virtual user" or user equivalent to the two actual A_1 and A_2 user points.

The coordinates x_v, y_v of the point V can be determined analytically by using

$$x_v = \frac{adx_1 - bcx_2 - bd(y_1 - y_2)}{ad - bc}$$
$$y_v = \frac{ady_2 - bcy_1 + ac(x_1 - x_2)}{ad - bc} \tag{14}$$

first calculating

$$a = c_1(y_1 - y_2) - rs_1(x_1 - x_2)$$
$$b = c_1(x_1 - x_2) + rs_1(y_1 - y_2)$$
$$c = c_2(y_1 - y_2) + rs_2(x_1 - x_2) \tag{15}$$
$$d = c_2(x_1 - x_2) - rs_2(y_1 - y_2)$$

where the symbols have the same meaning as that assumed in (9).

Clearly, the existence of the coordinates furnished by (14) is subordinate to verification of the conditions (6′) on the existence of the weight triangle.

Solution of the case $n > 3$

The case of networks connecting a generic number n of given points A_i can be reduced to the elementary case, already examined, of a network with just three user points.[10]

If we take as starting configurations the Steiner locally minimal topologies, the corresponding locally minimal network of the weighted problem can be located for every one. Applying the equivalence principle previously considered, each pair of user points linked to the same auxiliary point S_j can in fact be replaced by a single equivalent point V_j, whose coordinates can be calculated by using (14). The resulting transformed network is equivalent, in terms of cost, to the original one, but the number of its vertices is less than n.

By implementing similar successive transformations to this network a final network is achieved with just three user points and a single auxiliary junction coincident with one of the $(n - 2)$ original unknown auxiliary junctions.

Once this point has been located directly by using the formulas (7), we can proceed, using the same expressions, with the determination of the other unknown auxiliary junctions that appear in the transformed network from which the network being examined derives; tracing back through the various transformation stages previously implemented, we finally arrive at the original network configuration where the location of the last unknown auxiliary junctions can be completed.

Figure 4 gives a block diagram of the procedure described.

It should be pointed out that the procedure illustrated is able to afford the final solution if the conditions (4) and (6′) are respected before each application of (7) and (14). Should it happen, during the determination of an equivalent point V_j or an auxiliary point S_j that they are not satisfied and that therefore the point S_j in question is to be assumed coincident with a point A_i, the whole procedure must be begun again and applied to an initial network structure where the position of this auxiliary point, imposing the above coincidence, is known *a priori*.

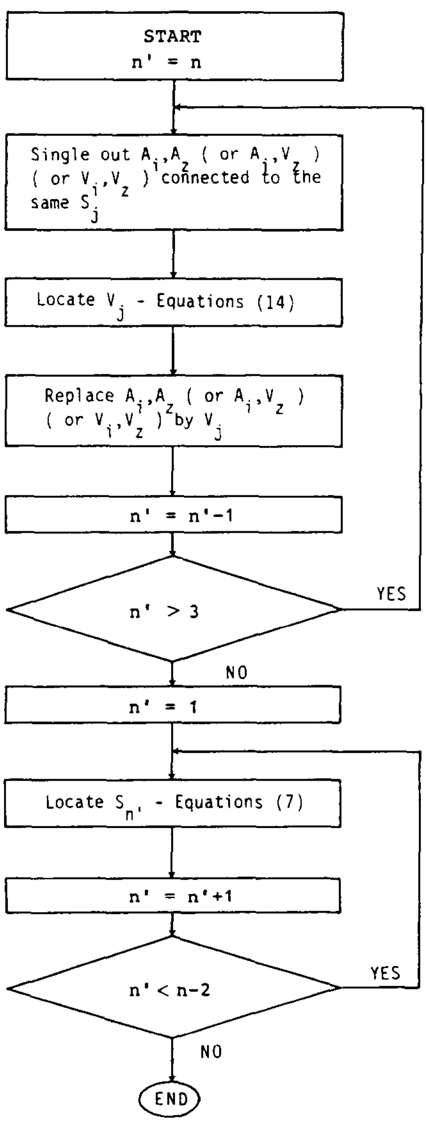

Figure 4. Block diagram of the solution algorithm for the case $n > 3$

Finally, it should be pointed out that to apply the algorithm proposed, the structure of the network to be optimized must be known *a priori*; hence it provides the minimum cost solution just relating to that structure. Thus to solve the optimization problem in an absolute sense, we must compare the various relative minimum costs obtained by applying the algorithm to all the possible network configurations interconnecting the n given points. It results that the computational onus increases exponentially with the number n of points.[10]

To reduce computational time, the reference topologies may be obtained either by means of devices based on a soap film analog,[11] which afford possible solutions directly, or with the available procedures for locating the minimal tree in the unweighted case, which resort to pruning methods, thereby minimizing the number of possible combinations.

Should the search be limited to very small subsets of the set of given points, then the problem could be more tractable. Tests carried out using Melzak's algorithm showed, for the unweighted case, that the shortest network for more than six randomly chosen points can, as a rule, be decomposed into minimal-length networks for smaller sets of points.

In the light of the foregoing the algorithm, in the case of large-scale networks, is clearly more fitting and efficient when applied to problems of local optimization.

Example of application

Figure 5 shows the numerical solutions obtained by a computer in the application of the proposed algorithm to a problem of local optimization of an electrical large-scale MV distribution network.

The objective is to find a minimum-cost network connecting $n = 4$ MV/LV stations (A_1, A_2, A_3, A_4) having the characteristics indicated in *Table 1*. It is assumed that the set of the four given users is supplied via the station A_1.

By introducing into the network $(n - 2) = 2$ auxiliary junction points (S_1, S_2), at most three different configurations (A, B, C) of the connection network can be identified, for each of which the most appropriate sections have been dimensioned[12] and the unit costs w of each branch estimated (see *Table 2*).

Figures 5(a), *5(b)*, and *5(c)* show the minimum-cost networks obtained for the three configurations A, B, C. For each of these the following are also indicated:

- the coordinates of the virtual user V_2 calculated by (14);
- the optimum coordinates of the auxiliary junction points S_1 and S_2 calculated by (7);
- the total cost of the optimum network relating to the configuration examined.

As can be observed, in the configuration C of *Figure 5(c)*, S_1, S_2, and A_3 coincide. This is a consequence of the fact that the second of the conditions (4) is not respected when the formulas (7) are applied to the triplet of users V_2, A_3, A_1 to find S_1 (which leads us to

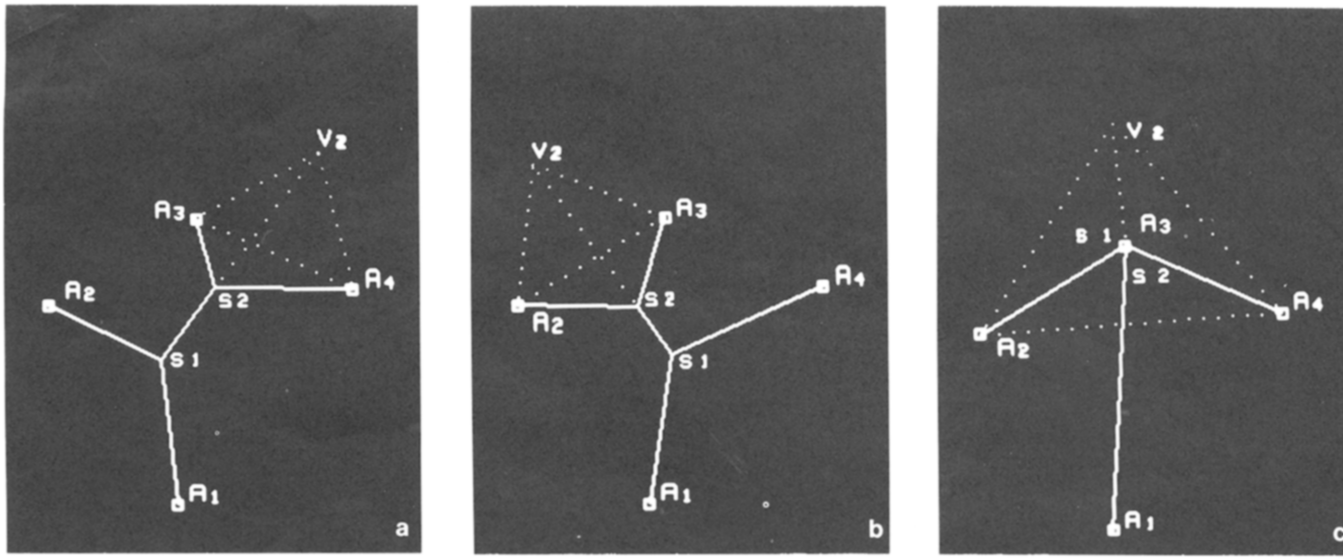

Figure 5. Minimal tree networks obtained in the sample application. (a) Co-ordinates—P_1: 50, 37; P_2: 32, 65; P_3: 53, 77; P_4: 75, 67; V_2: 70.7, 86.7; S_1: 47.7, 57.2; S_2: 55.6, 67.4; Total Cost = 90.83; (b) Co-ordinates—P_1: 50, 37; P_2: 32, 65; P_3: 53, 77; P_4: 75, 67; V_2: 34.5, 85; S_1: 53.5, 57.7; S_2: 48.8, 64.4; Total Cost = 92.73; (c) Co-ordinates—P_1: 50, 37; P_2: 32, 65; P_3: 53, 77; P_4: 75, 67; V_2: 52.2, 94.7; S_1: 53, 77; S_2: 53, 77; Total Cost = 104.51

Table 1. Users' demand

User points	Coordinates (X, Y)	Power (kVA)	Power factor ($\cos \varphi$)
A_1	(50, 37)	400	1
A_2	(32, 65)	500	1
A_3	(53, 77)	450	1
A_4	(73, 79)	550	1

Rated voltage: 15 kV

Table 2. Tree network configurations

Branch	Power (kVA)	Section (mm^2)	Unitary cost per unit, $w/w_{16\text{mm}^2}$
Network A			
S_1A_1	1500	35	1.4
S_1A_2	500	16	1
S_2A_3	450	16	1
S_2A_4	550	16	1
S_1S_2	1500	25	1.2
Network B			
S_1A_1	1500	35	1.4
S_1A_4	550	16	1
S_2A_2	500	16	1
S_2A_3	450	16	1
S_1S_2	950	25	1.2
Network C			
S_1A_1	1500	35	1.4
S_1A_3	450	16	1
S_2A_2	500	16	1
S_2A_4	550	16	1
S_1S_2	1050	25	1.2

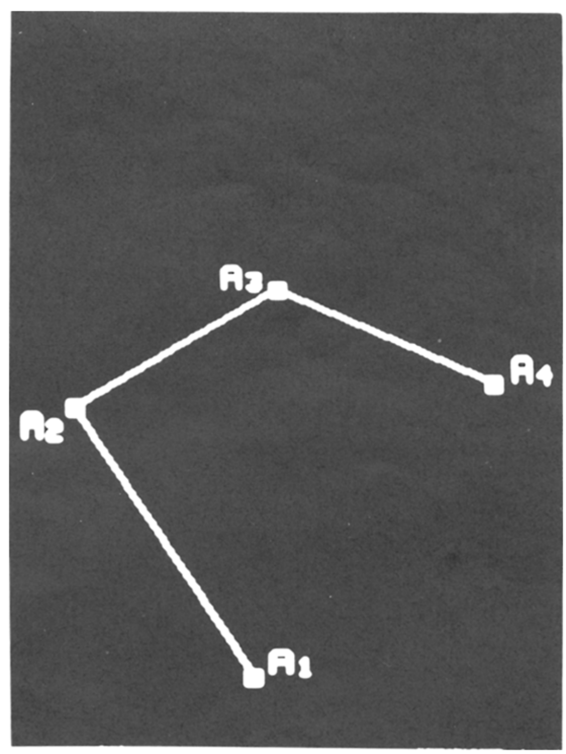

Figure 6. Minimal tree with direct links. Co-ordinates—P_1: 50, 37; P_2: 32, 65; P_3: 53, 77; P_4: 75, 67; Total Cost = 99.8

assume that this auxiliary point is the same as A_3) as well as to the triplet A_2, $S_1 = A_3$, A_4 for the determination of S_2 (which leads us to assume that S_2 also coincides with the user A_3).

A comparison of the three costs obtained clearly indicates that the network A is the optimum one.

If we disregard the extra cost of the auxiliary junctions introduced, the percentage savings derived by adopting the network A, compared to the optimum network with direct user links (see *Figure 6*), is about 9%.

Conclusion

In the present work a mathematical solution is provided in explicit form for the problem of optimum location of auxiliary junction points to be introduced into a tree connection network in view of minimizing total costs.

The result achieved, which derives from formulation of the problem in which link unit costs differ for each network branch, can be applied to various classes of networks (transport, communication, distribution, etc.) as well as to problems of location of plants or resources in the regional ambit.

The procedure described is particularly suitable for solving problems of local optimization of large tree networks.

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